Putnam Special Session: Real and Integral Inequalities

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Abstract

We'll give a short introduction to real number and integral inequalities, followed by a selection of training problems.

We'll begin with explaining convexity and then applying it for several famous inequalities. Next, we'll do integral inequalities. It is important to recognize the wide range of results gained from integral inequalities. First, most measure spaces one deals with are complete and σ-finite, so that theorems like Fubini-Tonelli (swapping the order of integration for double, triple, etc integrals) may be applied. Indeed, measure spaces on the Putnam are typically either probability spaces (where the measure of the whole space is 1, and measure is interpreted as probability), or $\mathbb{R}^n$ (where the measure is the Lebegue measure, and is interpreted as volume), which is σ-finite. Second, let X be a finite set, with probability P its measure. Integrals, A.K.A. expectations, are then actually just sums of values with coefficients that add up to 1. If Y is a random variable on X, then the expectation of Y is a *convex combination* of the values that Y takes. This gives integral inequalities double meaning, as they may be translated to summation inequalities on convex combinations. These summations need not necessarily be finite; indeed, integrals can also translate into infinite sequences.

Note that pictures of theorems are taken from the book *Classical Fourier Analysis*, 3rd edition, by Grafakos [Gra13]. Putnam problems are taken from the Putnam Archive [Ked].

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1 Convexity, Monotonicity, and Summation Inequalities

1.1 Convexity

Definition 1 (Convex Combination)**.** Let V be a vector space over $\mathbb{R}$, and let v_1, v_2 be two vectors. A *convex combination* of v_1 and v_2 is simply

$$tv_1 + (1-t)v_2,$$

where $t \in [0,1]$. If $t \in (0,1)$, we'll call these *strict convex combination* of v_1 and v_2.
Let $v_1, \ldots, v_n$ be vectors. Then a *convex combination* of them is simply

$$\sum_{i=1}^{n} a_n v_n,$$

where each $a_i \geq 0$, and

$$\sum_{i=1}^{n} a_n = 1.$$

If, in addition, $a_i > 0$ for every a_i, we again call this a *strict convex combination.*

Note that we can let $V = \mathbb{R}$, and the definitions remain the same. The following proposition is useful to easily induct on inequalities:

Proposition 1. *A convex combination of n vectors is simply $n-1$ iterated convex combinations on two vectors.*

It also naturally extends the definitions of the following objects (try to use this proposition to extend them yourself!):

Definition 2 (Convexity and Concavity)**.** Let V be a finite dimensional vector space over $\mathbb{R}$. A subset $S \subseteq V$ is called *convex* if the convex combination of any two vectors of S is in S. The convex hull of a set $A \subseteq V$ is the set of all convex combinations of elements from A. A function $f : S \to \mathbb{R}$ is called *convex* in a convex domain S if, for $x, y \in S$, $t \in [0,1]$

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

and is called *strictly convex* in a convex domain S if, for $x \neq y \in S$, $t \in (0,1)$,

$$f(tx + (1-t)y) < tf(x) + (1-t)y.$$

Similarly, a function is called *concave* and *strictly concave* if the previous inequalities are reversed.

Remark 1. (i) If $V = \mathbb{R}$, then subsets are convex iff they are intervals.

(ii) The convex hull of a set is convex. Convex hulls are typically used to define the creation of convex polyhedra, for optimization reasons, or for analytic purposes on normed linear spaces and complex analysis.

(iii) Some examples of convex sets in $\mathbb{R}^n$ are balls, cubes, prisms, convex polyhedra, and the sets lying below convex functions.

(iv) Some examples of famous convex functions are norms. Also, $|x|^p$ where $p \geq 1$, with strict convexity if $p > 1$.

(v) Some examples of famous strictly concave functions are $\ln x = \log x$ on $x > 0$, $|x|^p$ where $0 < p < 1$ on the domain $x > 0$.

(vi) Convex functions are bowls, and concave ones are frowns. You may stretch a string over the length of a convex function, and under the length of a concave function.

(vii) If $V = \mathbb{R}$, for double continuously differentiable functions f on an interval S, we get that a function is convex iff $\frac{d^2}{dx^2} f \geq 0$ on S and concave iff $\frac{d^2}{dx^2} f \leq 0$ on that set, with strict convexity/concavity following from strict inequalities. Convex functions lie above any tangent line, while concave functions lie below any tangent line.

(viii) It generally does not matter whether a convexity requirement is strict, as most problems deal with non-strict inequalities.

1.2 Consequences of Convexity (Tricks)

The following should be very natural to understand now:

Proposition 2. *Here are some common tricks:*

(i) $\ln x \leq x - 1$ *for* $x \geq 0$*, with equality iff* $x = 1$*.*

(ii) $\sqrt{1+x} - 1 \leq \frac{1}{2}x$ *for* $x \geq -1$*, with equality iff* $x = 0$*.*

(iii) Let $t \in [0,1]$*. Then* $|a|^t|b|^{1-t} \leq t|a| + (1-t)|b|$

(iv) AM-GM: Arithmetic Mean is greater than Geometric Mean.

$$\frac{1}{n}\sum_{i=1}^{n}|a_i| \geq \left(\prod_{i=1}^{n}|a_i|\right)^{1/n}$$

Proof. (i) Tangent line

(ii) Tangent line

(iii) Apply log on convex combination.

(iv) Apply log on convex combination.

□

Definition 3 (Holder Conjugates)**.** $-\infty < p, q \leq \infty$ are said to be *Holder conjugates* if

$$\frac{1}{p} + \frac{1}{q} = 1,$$

with

$$p' = \frac{p}{p-1}$$

denoting Holder conjugate for $p > 0$, and $1' = \infty, \infty' = 1$.

The above definition is convoluted at first, but it is very natural to apply convexity to it.

Proposition 3 (Young's Inequality of Products)**.** *Let* $1 < p < \infty$*. Then*

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}$$

Proof. Apply log on convex combination. □

1.3 A Note about Montonicity

Definition 4 (Monotonicity)**.** A function $f : S \to \mathbb{R}$ where $S \subseteq \mathbb{R}$ is said to be *increasing* if

$$\forall x \leq y \in S, f(x) \leq f(y),$$

and *strictly increasing* if

$$\forall x < y \in S, f(x) < f(y).$$

The definition of *decreasing* and *strictly decreasing* is similar, with the inequality sign reversed.

Remark 2. To check that a differentiable function is increasing on S, a sufficient condition is that $\frac{d}{dx}f \geq 0$ on S, with strict inequality being a sufficient (but not necessary) condition for strictly increasing. A similar process is done for decreasing.

For example, for $x \geq 0$, $\log x, x^p$ are strictly increasing. This can be used to transform certain inequalities to easier forms.

Example 1. Let $1 < p < \infty$. Show that $\exists b \geq 1$ so that

$$2^p \cdot \frac{\max((b-1)^{-1/2}, 1)^{1-p}}{b^p} > 1.$$

2 Common Norms and Quasi-norms on $\mathbb{R}^n$

2.1 Definition and Properties

Definition 5 (Seminorms, norms, and quasinorms)**.** Let V be a vector space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Consider the following properties of a function $\|\cdot\| : V \to \mathbb{R}$

(i) (Strict Positivity) $\|x\| = 0 \iff 0 = x \in V$

(ii) (Constant Multiplication) $\forall c \in \mathbb{K}, x \in V$, $\|cx\| = |c| \, \|x\|$

(iii) (Triangle Inequality, Subadditivity) $\forall x, y \in V$, $\|x + y\| \leq \|x\| + \|y\|$.

(iv) (Quasi-subadditivity) $\exists 1 < K < \infty$ s.t. $\forall x, y \in V, \|x + y\| \leq K(\|x\| + \|y\|)$.

We say that $\|\cdot\|$ is a *semi-norm* if it satisfies (ii) and (iii), a *norm* if it satisfies (i), (ii), and (iii), and a *quasi-norm* if it satisfies (i), (ii), (iv). Any norm is a semi-norm and a quasi-norm, but the converse is false.

Definition 6 (p-norms and p-quasi-norms)**.** Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $p \neq 0, \infty, -\infty$. We define

$$\|x\|_p := (|x_1|^p + |x_2|^p + \cdots + |x_n|^p))^{1/p}$$

and for $p = \infty$ we define

$$\|x\|_\infty = \sup(|x_1|, |x_2|, \ldots, |x_n|).$$

Finally, the definitions are naturally extended to infinite sums when $n = \infty$.

Theorem 1. *Let $x = (x_1, \ldots, x_n), y = (x_1, \ldots, y_n)$, and let xy be the pointwise product. Then*

(i) If $1 \leq p \leq \infty$, then $\|\cdot\|_p$ is a norm.

(ii) If $0 < p < 1$, then $\|\cdot\|_p$ is a quasi-norm with $K = 2^{\frac{1-p}{p}}$, which is the minimal such K.

(iii) (Holder's) For $1 \leq p \leq \infty$, $\|xy\|_1 \leq \|x\|_p \|y\|_{p'}$. If $p = 2$, this is Cauchy-Schwarz.

(iv) (Reverse Holder's) For $0 < p < 1$, if y has strictly positive components, then $\|xy\|_1 \geq \|x\|_p \|y\|_{p'}$.

(v) For $0 < p \leq 1$, $\|x\|_1 \leq \|x\|_p$

(vi) For $1 \leq p \leq \infty$, $\|x\|_1 \geq \|x\|_p$.

(vii) For $0 \leq p \leq 1$, $\|x\|_1 \geq n^{\frac{p-1}{p}} \|x\|_p$.

(viii) For $1 \leq p \leq \infty$, $\|x\|_1 \leq n^{\frac{p-1}{p}} \|x\|_p$.

Also, parts (i)-(vi) still work when $n = \infty$.

Proof. I recommend searching it up or asking AI. (Mostly, you need to apply convexity or concavity of $|\cdot|^p$, but Holder's and subadditivity will be more complicated and are textbook proofs.) □

If you haven't done much advanced "hard" analysis and norm equivalence, these may be hard to remember. Try to remember (i), (ii), (iii) ($p = 2$) at least.

2.2 Some Examples

Example 2. For positive numbers, prove that

$$3(a + b + c + d)^2 \geq 8(ab + ac + ad + bc + bd + cd).$$

Proof. Expand the LHS and subtract. You get

$$3(a^2 + b^2 + c^2 + d^2) \geq 2(ab + ac + ad + bc + bd + cd).$$

But if we write $x_1 = (a, b, c, d), x_2 = (b, c, d, a), x_3 = (c, d, a, b), x_4 = (d, a, b, c)$, we get that by Cauchy-Schwarz,

$$3\|x_1\|_2^2 = \|x_1\|_2 \|x_2\|_2 + \|x_1\|_2 \|x_3\|_2 + \|x_1\|_2 \|x_4\|_2 \geq \|x_1 x_2\|_1 + \|x_1 x_3\|_1 + \|x_1 x_4\|_1,$$

which is the inequality above. □

Sedrakyan's lemma – positive real numbers [edit]

Sedrakyan's inequality, also known as Bergström's inequality, Engel's form, Titu's lemma (or the T2 lemma), states that for real numbers $u_1, u_2, \ldots, u_n$ and positive real numbers $v_1, v_2, \ldots, v_n$:

$$\frac{(u_1 + u_2 + \cdots + u_n)^2}{v_1 + v_2 + \cdots + v_n} \leq \frac{u_1^2}{v_1} + \frac{u_2^2}{v_2} + \cdots + \frac{u_n^2}{v_n},$$

or, using summation notation,

$$\left(\sum_{i=1}^{n} u_i\right)^2 \Big/ \sum_{i=1}^{n} v_i \leq \sum_{i=1}^{n} \frac{u_i^2}{v_i}.$$

It is a direct consequence of the Cauchy–Schwarz inequality, obtained by using the dot product on $\mathbb{R}^n$ upon substituting $u_i' = \frac{u_i}{\sqrt{v_i}}$ and $v_i' = \sqrt{v_i}$. This form is especially helpful when the inequality involves fractions where the numerator is a perfect square.

Example 3.

3 Integral Inequalities

3.1 L^p Spaces

We'll introduce integral spaces so that much of Theorem 1 holds on those spaces. From now on, functions will be measurable and equality of functions will be with equality up to a set of measure 0 (almost everywhere, a.e.). This is a largely technical constraint that has no influence on problemsolving. From now on, let (X, μ) be our measure space. We assume that it is a complete measure space.

Definition 7 (L^p Spaces and L^p Norms)**.** Based on context, we can let $\mathbb{K} = \overline{\mathbb{R}}$ or $\mathbb{K} = \mathbb{C}$ where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$. Typically, we'll just choose $\mathbb{K} = \overline{\mathbb{R}}$, as that is more pertinent to problemsolving. If $p \neq 0, -\infty, \infty$, we can define for $f : X \to \mathbb{K}$

$$\|f\|_{L^p(X,\mu)} = \|f\|_{L^p} := \left(\int_X |f|^p d\mu\right)^{1/p},$$

(with the constraint that to be defined at $p < 0$, f is not 0 a.e.) and for $p = \infty$,

$$\|f\|_{L^\infty(X,\mu)} = \|f\|_{L^\infty} = \inf\{B > 0 : \mu(\{x \in X : |f(x)| > B\}) = 0\}.$$

Now, define for $0 < p \leq \infty$ (with functions equal if they are equal a.e.)

$$L^p(X, \mu) = L^p := \{f : X \to \mathbb{K} \text{ measurable} : \|f\|_{L^p} < \infty\}.$$

Theorem 2. *Let $f : X \to \mathbb{K}, g : X \to \mathbb{K}$, and let fg be the pointwise product. Let each $f_i : X \to \mathbb{K}$. Then*

(i) If $1 \leq p \leq \infty$, then $\|\cdot\|_{L^p}$ is a norm.

(ii) If $0 < p < 1$, then $\|\cdot\|_{L^p}$ is a quasi-norm with $K = 2^{\frac{1-p}{p}}$, which is the minimal such K.

(iii) (Completeness) If $p \geq 1$, then L^p is Banach, and if $0 < p < 1$, then L^p is quasi-Banach.

(iv) (Holder's) For $1 \leq p \leq \infty$, $\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^{p'}}$. If $p = 2$, this is Cauchy-Schwarz.

(v) (Reverse Holder's) For $0 < p < 1$, if g is nonzero a.e., then $\|fg\|_{L^1} \geq \|f\|_{L^p} \|g\|_{L^{p'}}$.

Proof. Look it up or ask AI. □

Remark 3. Convince yourself that the last 4 parts of Theorem 1 no longer hold (let $X = \mathbb{R}$ and $\mu = m$ typical Lebesgue measure, A.K.A. length).

Theorem 3 (Duals)**.** *See below.*

It is a well-known fact that the dual $(L^p)^*$ of L^p is isometric to $L^{p'}$ for all $1 \leq p < \infty$. Furthermore, the L^p norm of a function can be obtained via duality when $1 \leq p \leq \infty$ as follows:

$$\|f\|_{L^p} = \sup_{\|g\|_{L^{p'}}=1} \left| \int_X fg\, d\mu \right|.$$

3.2 Other Integral Inequalities

Proposition 4 (Limits and Jensen's)**.** *See below.*

1.1.3. Let (X, μ) be a measure space.
(a) If f is in $L^{p_0}(X, \mu)$ for some $p_0 < \infty$, prove that

$$\lim_{p \to \infty} \|f\|_{L^p} = \|f\|_{L^\infty}.$$

(b) (*Jensen's inequality*) Suppose that $\mu(X) = 1$. Show that

$$\|f\|_{L^p} \geq \exp\left(\int_X \log|f(x)|\,d\mu(x)\right)$$

for all $0 < p < \infty$.
(c) If $\mu(X) = 1$ and f is in some $L^{p_0}(X,\mu)$ for some $p_0 > 0$, then

$$\lim_{p\to 0}\|f\|_{L^p} = \exp\left(\int_X \log|f(x)|\,d\mu(x)\right)$$

with the interpretation $e^{-\infty} = 0$.

Remark 4. Part (b) actually says that if Y is a random variable on X and $\mathbb{E}$ denotes expectation (average), then

$$\mathbb{E}[\log|Y|] \leq \log\left(\mathbb{E}[|Y|^p]^{1/p}\right).$$

More generally, we have that if $\varphi : [0,\infty) \to \overline{\mathbb{R}}$ is concave, then

$$\mathbb{E}[\varphi|Y|] \leq \varphi\left(\mathbb{E}[|Y|]\right),$$

with the inequality reversed for convex functions.
Setting $\varphi = \cdot^2$, this is the inequality that variance is positive.
If X is a finite set, this reduces down to the definition of convexity/concavity. If X is a countably infinite set, this allows us to use a notion of "infinite" convex combinations. If X is something like $\mathbb{R}$, then this is a legitimate integral.

Proof. Use simple functions and concavity/convexity, then dominated convergence theorem. □

The next two are more technical:

Theorem 4 (Minkowski's Integral Inequality)**.** *See below.*

1.1.6. (a) (*Minkowski's integral inequality*) Let (X,μ) and (T,ν) be two σ-finite measure spaces and let $1 \leq p < \infty$. Show that for every nonnegative measurable function F on the product space $(X,\mu)\times(T,\nu)$ we have

$$\left[\int_T\left(\int_X F(x,t)\,d\mu(x)\right)^p d\nu(t)\right]^{\frac{1}{p}} \leq \int_X\left[\int_T F(x,t)^p\,d\nu(t)\right]^{\frac{1}{p}} d\mu(x),$$

(b) State and prove an analogous inequality when $p = \infty$.
(c) Prove that when $0 < p < 1$, then the preceding inequality is reversed.

Theorem 5 (Young's Inequality of Convolutions)**.** *See below.* $*$ *denotes convolution.*

Theorem 1.2.12. ***(Young's inequality)*** *Let* $1 \leq p,q,r \leq \infty$ *satisfy*

$$\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}. \tag{1.2.13}$$

Then for all f *in* $L^p(G)$ *and all* g *in* $L^r(G)$ *satisfying* $\|g\|_{L^r(G)} = \|\widetilde{g}\|_{L^r(G)}$ *we have* $f * g$ *exists* λ*-a.e. and satisfies*

$$\|f*g\|_{L^q(G)} \leq \|g\|_{L^r(G)}\|f\|_{L^p(G)}. \tag{1.2.14}$$

4 Chebychev's Inequality and Weak L^p

4.1 The Distribution Function

Definition 8 (The Distribution Function)**.** Let $f : X \to \overline{\mathbb{R}}$ measurable, and define for $\alpha \geq 0$:

$$d_f(\alpha) = \mu(\{x \in X : |f(x)| > \alpha\})$$

Proposition 5 (Integral Computations with $d_f(\alpha)$)**.** *See below.*

Proposition 1.1.4. *Let (X, μ) be a σ-finite measure space. Then for f in $L^p(X, \mu)$, $0 < p < \infty$, we have*

$$\|f\|_{L^p}^p = p \int_0^\infty \alpha^{p-1} d_f(\alpha)\, d\alpha. \tag{1.1.6}$$

Moreover, for any increasing continuously differentiable function φ on $[0, \infty)$ with $\varphi(0) = 0$ and every measurable function f on X with $\varphi(|f|)$ integrable on X, we have

$$\int_X \varphi(|f|)\, d\mu = \int_0^\infty \varphi'(\alpha) d_f(\alpha)\, d\alpha. \tag{1.1.7}$$

Proof. Indeed, we have

$$\begin{aligned} p \int_0^\infty \alpha^{p-1} d_f(\alpha)\, d\alpha &= p \int_0^\infty \alpha^{p-1} \int_X \chi_{\{x:\, |f(x)| > \alpha\}}\, d\mu(x)\, d\alpha \\ &= \int_X \int_0^{|f(x)|} p\alpha^{p-1}\, d\alpha\, d\mu(x) \\ &= \int_X |f(x)|^p\, d\mu(x) \\ &= \|f\|_{L^p}^p, \end{aligned}$$

where in the second equality we used Fubini's theorem, which requires the measure space to be σ-finite. This proves (1.1.6). Identity (1.1.7) follows similarly, replacing the function α^p by the more general function $\varphi(\alpha)$ which has similar properties. □

Theorem 6 (Chebychev's Inequality)**.** *Let $f \in L^p(X, \mu)$. Then*

$$d_f(\alpha) \leq \frac{\|f\|_{L^p}^p}{\alpha^p}$$

Remark 5. In terms of probability, if Y is a random variable on X, this translates to

$$P(|Y| > \alpha) \leq \frac{\mathbb{E}[|Y|]}{\alpha},$$

$$P(|Y - \mathbb{E}Y| > \alpha) \leq \frac{Var(Y)}{\alpha^2},$$

Proof.

$$\alpha^p d_f(\alpha) \leq \int_{\{x \in X : |f(x)| > \alpha\}} |f(x)|^p d\mu = \|f\|_{L^p}^p$$

□

4.2 Introduction to Weak L^p and Interpolation

Sometimes, we have functions that don't quite fit into L^p, but satisfy Chebychev's Inequality with a different constant. We call those the weak L^p spaces.

Definition 1.1.5. For $0 < p < \infty$, the space *weak* $L^p(X,\mu)$ is defined as the set of all μ-measurable functions f such that

$$\|f\|_{L^{p,\infty}} = \inf\left\{C > 0 : d_f(\alpha) \leq \frac{C^p}{\alpha^p} \quad \text{for all} \quad \alpha > 0\right\} \tag{1.1.8}$$

$$= \sup\left\{\gamma d_f(\gamma)^{1/p} : \gamma > 0\right\} \tag{1.1.9}$$

is finite. The space *weak* $L^\infty(X,\mu)$ is by definition $L^\infty(X,\mu)$.

One should check that (1.1.9) and (1.1.8) are in fact equal. The weak L^p spaces are denoted by $L^{p,\infty}(X,\mu)$. Two functions in $L^{p,\infty}(X,\mu)$ are considered equal if they are equal μ-a.e. The notation $L^{p,\infty}(\mathbf{R}^n)$ is reserved for $L^{p,\infty}(\mathbf{R}^n, |\cdot|)$. Using Proposition 1.1.3 (2), we can easily show that

$$\|kf\|_{L^{p,\infty}} = |k|\|f\|_{L^{p,\infty}}, \tag{1.1.10}$$

for any complex constant k. The analogue of (1.1.3) is

$$\|f+g\|_{L^{p,\infty}} \leq c_p\left(\|f\|_{L^{p,\infty}} + \|g\|_{L^{p,\infty}}\right), \tag{1.1.11}$$

where $c_p = \max(2, 2^{1/p})$, a fact that follows from Proposition 1.1.3 (3), taking both α and β equal to $\alpha/2$. We also have that

$$\|f\|_{L^{p,\infty}(X,\mu)} = 0 \Rightarrow f = 0 \qquad \mu\text{-a.e.} \tag{1.1.12}$$

In view of (1.1.10), (1.1.11), and (1.1.12), $L^{p,\infty}$ is a *quasi-normed linear space* for $0 < p < \infty$.

The weak L^p spaces are larger than the usual L^p spaces. We have the following:

Proposition 1.1.6. *For any $0 < p < \infty$ and any f in $L^p(X,\mu)$ we have*

$$\|f\|_{L^{p,\infty}} \leq \|f\|_{L^p}.$$

Hence the embedding $L^p(X,\mu) \subseteq L^{p,\infty}(X,\mu)$ holds.

Proof. This is just a trivial consequence of Chebyshev's inequality:

Proposition 1.1.14. *Let $0 < p < q \leq \infty$ and let f in $L^{p,\infty}(X,\mu) \cap L^{q,\infty}(X,\mu)$, where X is a σ-finite measure space. Then f is in $L^r(X,\mu)$ for all $p < r < q$ and*

$$\|f\|_{L^r} \leq \left(\frac{r}{r-p} + \frac{r}{q-r}\right)^{\frac{1}{r}} \|f\|_{L^{p,\infty}}^{\frac{\frac{1}{r}-\frac{1}{q}}{\frac{1}{p}-\frac{1}{q}}} \|f\|_{L^{q,\infty}}^{\frac{\frac{1}{p}-\frac{1}{r}}{\frac{1}{p}-\frac{1}{q}}}, \tag{1.1.22}$$

with the interpretation that $1/\infty = 0$.

Theorem 1.3.2. *Let (X,μ) be a σ-finite measure space, let (Y,ν) be another measure space, and let $0 < p_0 < p_1 \leq \infty$. Let T be a sublinear operator defined on $L^{p_0}(X)+L^{p_1}(X) = \{f_0+f_1 : f_j \in L^{p_j}(X_j), j=0,1\}$ and taking values in the space of measurable functions on Y. Assume that there exist $A_0, A_1 < \infty$ such that*

$$\|T(f)\|_{L^{p_0,\infty}(Y)} \leq A_0 \|f\|_{L^{p_0}(X)} \qquad \text{for all } f \in L^{p_0}(X), \tag{1.3.5}$$

$$\|T(f)\|_{L^{p_1,\infty}(Y)} \leq A_1 \|f\|_{L^{p_1}(X)} \qquad \text{for all } f \in L^{p_1}(X). \tag{1.3.6}$$

Then for all $p_0 < p < p_1$ and for all f in $L^p(X)$ we have the estimate

$$\|T(f)\|_{L^p(Y)} \leq A \|f\|_{L^p(X)}, \tag{1.3.7}$$

where

$$A = 2\left(\frac{p}{p-p_0} + \frac{p}{p_1-p}\right)^{\frac{1}{p}} A_0^{\frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{p_0}-\frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}}. \tag{1.3.8}$$

5 Miscellaneous

5.1 A-priori Estimates

This one is from Guermond's lecture notes.

A very common tactic in a-priori estimates it the following. Suppose that we have function u on one side of an equation and $\partial_x u$ (first derivative) on the other. Multiplying by $\partial_x u$, we have $\partial_x u \cdot u = \frac{1}{2}\partial_x(u^2)$ on one side and $(\partial_x u)^2$ on the other. Integrating, we get an estimate of the L^2 norm of $\partial_x u$ based on the boundary values of u.

Lemma 6.5 (1D Poincaré-Steklov). *Let $\ell_D > 0$ and let $D := (0, \ell_D)$. The following holds for all functions $u : D \to \mathbb{R}$ such that $\|u\|_{L^2(D)} < \infty$, $\|\partial_x u\|_{L^2(D)} < \infty$:*

$$\sqrt{2}\ell_D^{-1}\|u\|_{L^2(D)} \le \|\partial_x u\|_{L^2(D)} \quad \text{if } u(0) = 0 \tag{6.5a}$$

$$\sqrt{8}\ell_D^{-1}\|u\|_{L^2(D)} \le \|\partial_x u\|_{L^2(D)} \quad \text{if } u(0) = 0 \text{ and } u(\ell_D) = 0. \tag{6.5b}$$

Proof. (i) We make the proof assuming $u \in C^1(D;\mathbb{R})$. The complete proof assuming only $\|u\|_{L^2(D)} < \infty$, $\|\partial_x u\|_{L^2(D)} < \infty$ is out of the scope of the book. Since that $u(0) = 0$, the fundamental theorem of calculus implies that $u(x) = \int_0^x u(\xi)\,\mathrm{d}\xi$ for all $x \in D$. Hence,

$$\begin{aligned}|u(x)| &= \left|\int_0^x \partial_\xi u(\xi)\,\mathrm{d}\xi\right| \le \int_0^x |\partial_\xi u(\xi)|\,\mathrm{d}\xi = \int_0^x 1\times|\partial_\xi u(\xi)|\,\mathrm{d}\xi \\ &\le \left(\int_0^x 1\,\mathrm{d}\xi\right)^{\frac{1}{2}}\left(\int_0^x (\partial_\xi u(\xi))^2\,\mathrm{d}t\right)^{\frac{1}{2}} \le x^{\frac{1}{2}}\|\partial_x u\|_{L^2(D)}.\end{aligned}$$

Integrating over D, this gives $\|u\|^2_{L^2(D)} \le (\int_0^{\ell_D} x\,\mathrm{d}x)\|\partial_x u\|^2_{L^2(D)} = \frac{\ell_D}{2}\|\partial_x u\|^2_{L^2(D)}$. This proves the first inequality.

(ii) Assume now that $u(0) = 0$ and $u(\ell_D) = 0$, i.e., $u_{|\partial D} = 0$. If $x \in (0, \frac{\ell_D}{2})$, the above argument shows that $|u(x)| \le x^{\frac{1}{2}}\|\partial_x u\|_{L^2(0,\frac{\ell_D}{2})}$. Similarly, if $x \in (\frac{\ell_D}{2}, \ell_D)$, we have $|u(x)| \le (\ell_D - x)^{\frac{1}{2}}\|\partial_x u\|_{L^2(\frac{\ell_D}{2},\ell_D)}$. We then infer that

$$\begin{aligned}\|u\|^2_{L^2(D)} &= \int_0^{\frac{\ell_D}{2}} u(x)^2\,\mathrm{d}x + \int_{\frac{\ell_D}{2}}^{\ell_D} u(x)^2\,\mathrm{d}x \\ &\le \|\partial_x u\|^2_{L^2(0,\frac{\ell_D}{2})}\int_0^{\frac{\ell_D}{2}} x\,\mathrm{d}x + \|\partial_x u\|^2_{L^2(\frac{\ell_D}{2},\ell_D)}\int_{\frac{1}{2}}^{\ell_D}(\ell_D - x)\,\mathrm{d}x \\ &\le \frac{\ell_D}{8}\left(\|\partial_x u\|^2_{L^2(0,\frac{\ell_D}{2})} + \|\partial_x u\|^2_{L^2(\frac{\ell_D}{2},\ell_D)}\right) = \frac{\ell_D}{8}\|\partial_x u\|^2_{L^2(0,\ell_D)}.\end{aligned}$$

This proves the second inequality. □

Theorem 6.6 (Poincaré-Steklov inequality). *Let D be an open, bounded, and connected domain in $\mathbb{R}^d$. Let ℓ_D denote the diameter of D. There exists a constant c_{ps} (called Poincaré-Steklov constant) that only depends on the shape of D so that*

$$c_{\mathsf{ps}}\ell_D^{-2}\int_D u(\boldsymbol{x})^2\,\mathrm{d}x \le \int_D \|\nabla u(\boldsymbol{x})\|^2_{\ell^2}\,\mathrm{d}x, \tag{6.6}$$

for every differentiable function u s.t. $\int_D \|\nabla u(\boldsymbol{x})\|^2_{\ell^2}\,\mathrm{d}x < \infty$ and $u_{|\partial D} = 0$.

6 Exercises

In no particular order:

Question (1995A2). *(note: having a converging improper integral does NOT imply the function is in L^1. However, the converse is true.)*

A–2 For what pairs (a,b) of positive real numbers does the improper integral

$$\int_b^\infty \left(\sqrt{\sqrt{x+a}-\sqrt{x}}-\sqrt{\sqrt{x}-\sqrt{x-b}}\right)dx$$

converge?

Question (2003A2). .

A2 Let $a_1,a_2,\ldots,a_n$ and $b_1,b_2,\ldots,b_n$ be nonnegative real numbers. Show that

$$(a_1a_2\cdots a_n)^{1/n}+(b_1b_2\cdots b_n)^{1/n}$$
$$\leq [(a_1+b_1)(a_2+b_2)\cdots(a_n+b_n)]^{1/n}.$$

Question (2004A6). .

A6 Suppose that $f(x,y)$ is a continuous real-valued function on the unit square $0\leq x\leq 1, 0\leq y\leq 1$. Show that

$$\int_0^1\left(\int_0^1 f(x,y)dx\right)^2 dy+\int_0^1\left(\int_0^1 f(x,y)dy\right)^2 dx$$
$$\leq \left(\int_0^1\int_0^1 f(x,y)dxdy\right)^2+\int_0^1\int_0^1 [f(x,y)]^2 dxdy.$$

Question (2004B2). .

B2 Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}}<\frac{m!}{m^m}\frac{n!}{n^n}.$$

Question (2010B1). .

B1 Is there an infinite sequence of real numbers $a_1, a_2, a_3, \ldots$ such that

$$a_1^m + a_2^m + a_3^m + \cdots = m$$

for every positive integer m?

Question (2020B5). .

B5 For $j \in \{1,2,3,4\}$, let z_j be a complex number with $|z_j| = 1$ and $z_j \neq 1$. Prove that

$$3 - z_1 - z_2 - z_3 - z_4 + z_1 z_2 z_3 z_4 \neq 0.$$

Question (2025A2). .

A2 Find the largest real number a and the smallest real number b such that

$$ax(\pi - x) \leq \sin x \leq bx(\pi - x)$$

for all x in the interval $[0, \pi]$.

Question (2025B2). .

B2 Let $f\colon [0,1] \to [0,\infty)$ be strictly increasing and continuous. Let R be the region bounded by $x = 0$, $x = 1$, $y = 0$, and $y = f(x)$. Let x_1 be the x-coordinate of the centroid of R. Let x_2 be the x-coordinate of the centroid of the solid generated by rotating R around the x-axis. Prove that $x_1 < x_2$.

Question (2016B1). *(note: at least prove the convergence.)*

B1 Let $x_0, x_1, x_2, \ldots$ be the sequence such that $x_0 = 1$ and for $n \geq 0$,

$$x_{n+1} = \ln(e^{x_n} - x_n)$$

(as usual, the function ln is the natural logarithm). Show that the infinite series

$$x_0 + x_1 + x_2 + \cdots$$

converges and find its sum.

Question (2016A6). .

A6 Find the smallest constant C such that for every real polynomial $P(x)$ of degree 3 that has a root in the interval $[0,1]$,

$$\int_0^1 |P(x)|\,dx \le C \max_{x\in[0,1]} |P(x)|.$$

Question (1993B4). .

B–4 The function $K(x,y)$ is positive and continuous for $0 \le x \le 1, 0 \le y \le 1$, and the functions $f(x)$ and $g(x)$ are positive and continuous for $0 \le x \le 1$. Suppose that for all x, $0 \le x \le 1$,

$$\int_0^1 f(y)K(x,y)\,dy = g(x)$$

and

$$\int_0^1 g(y)K(x,y)\,dy = f(x).$$

Show that $f(x) = g(x)$ for $0 \le x \le 1$.

Question (2007B2). .

B–2 Suppose that $f : [0,1] \to \mathbb{R}$ has a continuous derivative and that $\int_0^1 f(x)\,dx = 0$. Prove that for every $\alpha \in (0,1)$,

$$\left|\int_0^\alpha f(x)\,dx\right| \le \frac{1}{8} \max_{0\le x\le 1} |f'(x)|.$$

Question (1996B2). .

B–2 Show that for every positive integer n,

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1\cdot 3\cdot 5\cdots(2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}.$$

Question (2021B2). .

B2 Determine the maximum value of the sum

$$S = \sum_{n=1}^{\infty} \frac{n}{2^n} (a_1 a_2 \cdots a_n)^{1/n}$$

over all sequences $a_1, a_2, a_3, \cdots$ of nonnegative real numbers satisfying

$$\sum_{k=1}^{\infty} a_k = 1.$$

Question (2021B6). .

B6 Given an ordered list of $3N$ real numbers, we can *trim* it to form a list of N numbers as follows: We divide the list into N groups of 3 consecutive numbers, and within each group, discard the highest and lowest numbers, keeping only the median.

Consider generating a random number X by the following procedure: Start with a list of 3^{2021} numbers, drawn independently and uniformly at random between 0 and 1. Then trim this list as defined above, leaving a list of 3^{2020} numbers. Then trim again repeatedly until just one number remains; let X be this number. Let μ be the expected value of $|X - \frac{1}{2}|$. Show that

$$\mu \geq \frac{1}{4} \left(\frac{2}{3} \right)^{2021}.$$

Question (2020B6). .

B6 Let n be a positive integer. Prove that

$$\sum_{k=1}^{n} (-1)^{\lfloor k(\sqrt{2}-1) \rfloor} \geq 0.$$

(As usual, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

Question (2020A6). .

A6 For a positive integer N, let f_N[1] be the function defined by

$$f_N(x) = \sum_{n=0}^{N} \frac{N+1/2-n}{(N+1)(2n+1)} \sin((2n+1)x).$$

Determine the smallest constant M such that $f_N(x) \leq M$ for all N and all real x.

Question (2019A6). .

A6 Let g be a real-valued function that is continuous on the closed interval $[0,1]$ and twice differentiable on the open interval $(0,1)$. Suppose that for some real number $r > 1$,

$$\lim_{x \to 0^+} \frac{g(x)}{x^r} = 0.$$

Prove that either

$$\lim_{x \to 0^+} g'(x) = 0 \qquad \text{or} \qquad \limsup_{x \to 0^+} x^r |g''(x)| = \infty.$$

7 Tips

1. Move everything into one integral. (Linearity of integration, Fubini-Tonelli theorem.)
2. Use Holder's Inequality to change exponents, thereby making polynomials factorizable.
3. Write sequences as vectors in different ways, abusing Holder's.
4. Plug in values, trying to find maxima/minima.
5. Induction.
6. Riemann sums / simple functions / Simpson's Rule.
7. When it's integers, consider using combinatorics instead.
8. Big O notation and Taylor expansions (see: basic physics estimates).
9. Check your algebra, seeing if you missed important terms. (Typically, if the resulting inequality is trivial, you probably did some bad algebra.)
10. Draw graphs.
11. Squares are positive.
12. Tangent lines.
13. Min/max with Calc I principles. If there's an inequality where a supremum (maximum) can be made, pass to supremum (maximum).
14. Reduce to smaller problems if there's a lot of variables.
15. The constant $\frac{1}{8}$ is gotten by integrating x over half of an interval. Meanwhile, the constant $\frac{1}{2}$ is gotten by integrating x over the full interval.
16. What happens when a function is constant or close to constant? Nonnegative? Strictly positive?
17. If functions are not zero, you are allowed to divide. Don't forget about operations that are not addition or absolute values.
18. Combine terms sometimes. This may help with the next tip:
19. You want to reduce the number of times you use inequality estimates. That is, you should have less inequality signs in your inequality, to avoid making too weak of an estimate.
20. Symmetry arguments.
21. Suppose that you want to prove a bound on $Q \geq 0$. Raising to the nth power, suppose you have the inequality
$$Q^n \leq f(n).$$
Then we have
$$Q \leq f(n)^{1/n}.$$
We know therefore that
$$Q \leq \inf_n (f(n)^{1/n}) \leq \lim_{n\to\infty} (f(n)^{1/n}).$$
22. Sometimes, you just have to compute :(

References

[Gra13] Loukas Grafakos. *Classical Fourier Analysis*. 3rd ed. Springer, 2013.

[Ked] Kiran Kedlaya. *Putnam Archive*. URL: https://kskedlaya.org/putnam-archive/.