Putnam Special Session: Matrix Combinatorics

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Abstract

Putnam problems very often involve enumerating the number of matrices with certain properties, or performing game or group-like operations on matrices, or using matrices to solve combinatorial problems. We give relevant exercises from past Putnams and some additional tips to help with these kinds of problems.

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1 Exercises

Question (1986A4). .

A–4 A *transversal* of an $n \times n$ matrix A consists of n entries of A, no two in the same row or column. Let $f(n)$ be the number of $n \times n$ matrices A satisfying the following two conditions:

(a) Each entry $a_{i,j}$ of A is in the set $\{-1, 0, 1\}$.

(b) The sum of the n entries of a transversal is the same for all transversals of A.

An example of such a matrix A is

$$A = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Determine with proof a formula for $f(n)$ of the form

$$f(n) = a_1 b_1^n + a_2 b_2^n + a_3 b_3^n + a_4,$$

where the a_i's and b_i's are rational numbers.

Question (1992B6). .

B–6 Let $\mathscr{M}$ be a set of real $n \times n$ matrices such that

(i) $I \in \mathscr{M}$, where I is the $n \times n$ identity matrix;

(ii) if $A \in \mathscr{M}$ and $B \in \mathscr{M}$, then either $AB \in \mathscr{M}$ or $-AB \in \mathscr{M}$, but not both;

(iii) if $A \in \mathscr{M}$ and $B \in \mathscr{M}$, then either $AB = BA$ or $AB = -BA$;

(iv) if $A \in \mathscr{M}$ and $A \neq I$, there is at least one $B \in \mathscr{M}$ such that $AB = -BA$.

Prove that $\mathscr{M}$ contains at most n^2 matrices.

Question (2011A4). .

A4 For which positive integers n is there an $n \times n$ matrix with integer entries such that every dot product of a row with itself is even, while every dot product of two different rows is odd?

Question (2018A2). .

A2 Let $S_1, S_2, \ldots, S_{2^n-1}$ be the nonempty subsets of $\{1,2,\ldots,n\}$ in some order, and let M be the $(2^n-1) \times (2^n-1)$ matrix whose (i,j) entry is

$$m_{ij} = \begin{cases} 0 & \text{if } S_i \cap S_j = \emptyset; \\ 1 & \text{otherwise.} \end{cases}$$

Calculate the determinant of M.

Question (2021B5). .

B5 Say that an n-by-n matrix $A = (a_{ij})_{1 \le i,j \le n}$ with integer entries is *very odd* if, for every nonempty subset S of $\{1,2,\ldots,n\}$, the $|S|$-by-$|S|$ submatrix $(a_{ij})_{i,j \in S}$ has odd determinant. Prove that if A is very odd, then A^k is very odd for every $k \ge 1$.

Question (2023B6). .

B6 Let n be a positive integer. For i and j in $\{1,2,\ldots,n\}$, let $s(i,j)$ be the number of pairs (a,b) of nonnegative integers satisfying $ai + bj = n$. Let S be the n-by-n matrix whose (i,j) entry is $s(i,j)$. For example, when $n = 5$, we have $S = \begin{bmatrix} 6 & 3 & 2 & 2 & 2 \\ 3 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 & 2 \end{bmatrix}$. Compute the determinant of S.

Question (2024B1). .

B1 Let n and k be positive integers. The square in the ith row and jth column of an n-by-n grid contains the number $i + j - k$. For which n and k is it possible to select

n squares from the grid, no two in the same row or column, such that the numbers contained in the selected squares are exactly $1, 2, \ldots, n$?

Question (2025B4). .

B4 For $n \geq 2$, let $A = [a_{i,j}]_{i,j=1}^{n}$ be an n-by-n matrix of nonnegative integers such that

(a) $a_{i,j} = 0$ when $i + j \leq n$;

(b) $a_{i+1,j} \in \{a_{i,j}, a_{i,j}+1\}$ when $1 \leq i \leq n-1$ and $1 \leq j \leq n$; and

(c) $a_{i,j+1} \in \{a_{i,j}, a_{i,j}+1\}$ when $1 \leq i \leq n$ and $1 \leq j \leq n-1$.

Let S be the sum of the entries of A, and let N be the number of nonzero entries of A. Prove that

$$S \leq \frac{(n+2)N}{3}.$$

B1 Consider an m-by-n grid of unit squares, indexed by (i, j) with $1 \leq i \leq m$ and $1 \leq j \leq n$. There are $(m-1)(n-1)$ coins, which are initially placed in the squares (i, j) with $1 \leq i \leq m-1$ and $1 \leq j \leq n-1$. If a coin occupies the square (i, j) with $i \leq m-1$ and $j \leq n-1$ and the squares $(i+1, j)$, $(i, j+1)$, and $(i+1, j+1)$ are unoccupied, then a legal move is to slide the coin from (i, j) to $(i+1, j+1)$. How many distinct configurations of coins can be reached starting from the initial configuration by a (possibly empty) sequence of legal moves?

Question (2023B1).

Question (1995B3). .

B–3 To each positive integer with n^2 decimal digits, we associate the determinant of the matrix obtained by writing the digits in order across the rows. For example, for $n = 2$, to the integer 8617 we associate $\det \begin{pmatrix} 8 & 6 \\ 1 & 7 \end{pmatrix} = 50$. Find, as a function of n, the sum of all the determinants associated with n^2-digit integers. (Leading digits are assumed to be nonzero; for example, for $n = 2$, there are 9000 determinants.)

Question (1996A3). .

A–3 Suppose that each of 20 students has made a choice of anywhere from 0 to 6 courses from a total of 6 courses offered. Prove or disprove: there are 5 students and 2 courses such that all 5 have chosen both courses or all 5 have chosen neither course.

Question (2005A4). .

A4 Let H be an $n \times n$ matrix all of whose entries are ± 1 and whose rows are mutually orthogonal. Suppose H has an $a \times b$ submatrix whose entries are all 1. Show that $ab \leq n$.

Question (2002A4). .

A4 In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3×3 matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the 3×3 matrix is completed with five 1's and four 0's. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?

Question (2016B4). .

B4 Let A be a $2n \times 2n$ matrix, with entries chosen independently at random. Every entry is chosen to be 0 or 1, each with probability $1/2$. Find the expected value of $\det(A - A^t)$ (as a function of n), where A^t is the transpose of A.

Question (2006B4). .

B–4 Let Z denote the set of points in $\mathbb{R}^n$ whose coordinates are 0 or 1. (Thus Z has 2^n elements, which are the vertices of a unit hypercube in $\mathbb{R}^n$.) Given a vector subspace V of $\mathbb{R}^n$, let $Z(V)$ denote the number of members of Z that lie in V. Let k be given, $0 \leq k \leq n$. Find the maximum, over all vector subspaces $V \subseteq \mathbb{R}^n$ of dimension k, of the number of points in $V \cap Z$. [Editorial note: the proposers probably intended to write $Z(V)$ instead of "the number of points in $V \cap Z$", but this changes nothing.]

Question (2008B6). .

B6 Let n and k be positive integers. Say that a permutation σ of $\{1,2,\ldots,n\}$ is *k-limited* if $|\sigma(i) - i| \leq k$ for all i. Prove that the number of k-limited permutations of $\{1,2,\ldots,n\}$ is odd if and only if $n \equiv 0$ or $1 \pmod{2k+1}$.

Question (2011B4). .

B4 In a tournament, 2011 players meet 2011 times to play a multiplayer game. Every game is played by all 2011 players together and ends with each of the players either winning or losing. The standings are kept in two 2011×2011 matrices, $T = (T_{hk})$ and $W = (W_{hk})$. Initially, $T = W = 0$. After every game, for every (h,k) (including for $h = k$), if players h and k tied (that is, both won or both lost), the entry T_{hk} is increased by 1, while if player h won and player k lost, the entry W_{hk} is increased by 1 and W_{kh} is decreased by 1.

Prove that at the end of the tournament, $\det(T + iW)$ is a non-negative integer divisible by 2^{2010}.

Question (2012A5). .

A5 Let $\mathbb{F}_p$ denote the field of integers modulo a prime p, and let n be a positive integer. Let v be a fixed vector in $\mathbb{F}_p^n$, let M be an $n \times n$ matrix with entries of $\mathbb{F}_p$, and define $G : \mathbb{F}_p^n \to \mathbb{F}_p^n$ by $G(x) = v + Mx$. Let $G^{(k)}$ denote the k-fold composition of G with itself, that is, $G^{(1)}(x) = G(x)$ and $G^{(k+1)}(x) = G(G^{(k)}(x))$. Determine all pairs p,n for which there exist v and M such that the p^n vectors $G^{(k)}(0)$, $k = 1,2,\ldots,p^n$ are distinct.

Question (2014B5). .

B5 In the 75th annual Putnam Games, participants compete at mathematical games. Patniss and Keeta play a game in which they take turns choosing an element from the group of invertible $n \times n$ matrices with entries in the field $\mathbb{Z}/p\mathbb{Z}$ of integers modulo p, where n is a fixed positive integer and p is a fixed prime number. The rules of the game are:

(1) A player cannot choose an element that has been chosen by either player on any previous turn.

(2) A player can only choose an element that commutes with all previously chosen elements.

(3) A player who cannot choose an element on his/her turn loses the game.

Patniss takes the first turn. Which player has a winning strategy? (Your answer may depend on n and p.)

Question (2014A6). .

A6 Let n be a positive integer. What is the largest k for which there exist $n \times n$ matrices $M_1,\ldots,M_k$ and $N_1,\ldots,N_k$ with real entries such that for all i and j, the matrix product M_iN_j has a zero entry somewhere on its diagonal if and only if $i \neq j$?

Question (2025A4). .

A4 Find the minimal value of k such that there exist k-by-k real matrices $A_1, \ldots, A_{2025}$ with the property that $A_i A_j = A_j A_i$ if and only if $|i - j| \in \{0, 1, 2024\}$.

2 Tips

1. Common combinatorial objects to note: perfect matchings, lattice paths, tilings, surjections, injections, functions, set partitions, compositions, weak compositions, integer partitions, choosing, hypergeometric, words, permutations, permutation avoidance. See: twelve-fold way.
2. Number theory (prime factorizations) can sometimes be of use.
3. Try smaller cases first, to find a pattern.
4. $\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$.
5. Compute determinants by row/column manipulation or straight up expansion.
6. Encode combinatorial objects as vectors.
7. For finite dimension, $AB = I \implies BA = I$.
8. Try to convert combinatorics into linear algebra (meaning: MATRICES) if possible. Sometimes it's not possible.
9. What does each row/column represent? Each entry represent?
10. What will change if you permute rows/columns?
11. What is the smallest local constraint? Is there an adjacency/incidence matrix?
12. Mobius inversions and basis changes.
13. Try to find eigenvectors by guessing really well.
14. Expectation is linear, and independence kills most terms.
15. Reduction on modular arithmetic / finite fields. $\mathbb{Z}_2$ is your best bet, usually, unless you're working modulo some other number.
16. If it's an A3/B3 or above and it was too easy, then you probably did it wrong.
17. Try summing over things.
18. Block matrices.
19. An integer matrix which is symmetric can be thought of as a graph incidence matrix. Meanwhile, an integer matrix which is antisymmetric can be thought of as a diagraph incidence matrix. You can try to interpret other matrices similarly in case-by-case basis.
20. If looking for a contradiction, maximal/minimal cases are good.
21. n even vs. n odd.
22. Work on simple/symmetric examples first.