Pyranim: A Combinatorial Game of Doom and Despair

TX-LA Undergraduate Math Conference @ LSU

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Apr 11, 2026

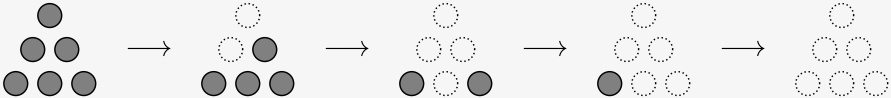
Pyranim

Setup

Pyranim is a Nim-inspired two-player impartial non-random game played on an n-tier triangular stack of chips. On each turn, a player removes a *visible* chip on the outer boundary; i.e. a chip without two chips that are on the row directly above it. Any chips that are no longer supported by two chips below are removed with it. The player who takes the final chip wins, or alternatively, the player who can no longer move loses.

Example 1

Example game sequence (the second player wins):



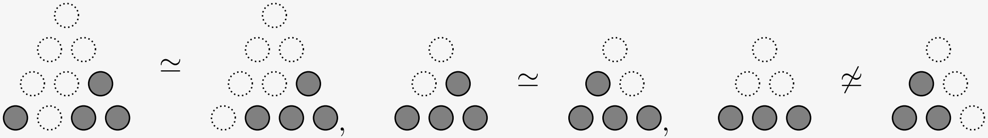
Isomorphic States

Definition 2

Let X and Y be two states. Then X and Y are **isomorphic** ($X \simeq Y$), if, given that the next legal moves from X are $\{X_1, \ldots, X_m\}$ and similarly for Y, then there is a bijection $\varphi : \{X_1, \ldots, X_m\} \longleftrightarrow \{Y_1, \ldots, Y_m\}$ such that $\forall i \in [m], \varphi(X_i) \simeq X_i$.

This defines a non-trivial equivalence relation on the game states. At a single time, we only consider the isomorphisms that result from the n-pyramid.

Example 3



Research Questions

Game theory:

- How does this game relate to previously-studied games?
- Which player has a winning strategy? What strategy ensures their success?
- What is the *nimber* value of each n-pyramid? Of every state?

Combinatorics:

- What classical combinatorial objects encode the state space?
- How many non-isomorphic states are there resulting from an n-pyramid?
- What is the distribution of non-isomorophic states resulting from an n-pyramid?

Variable Definitions

- P_n: the nimber of the n-pyramid.
- $\mathcal{X}_n$: the set of all possible game states that can be reached from the n-pyramid.
- $C_{n+1} = |\mathcal{X}_n|$ (the $(n+1)$th Catalan number).
- K_n: the number of isomorphism equivalence classes of $\mathcal{X}_n$.
- $M_{n,k}$: the number of equivalence classes in $\mathcal{X}_n$ that have k chips.
- $M_n = \max_k M_{n,k}$ for fixed n.
- L_n: the minimal k for which $M_{n,k} = M_n$

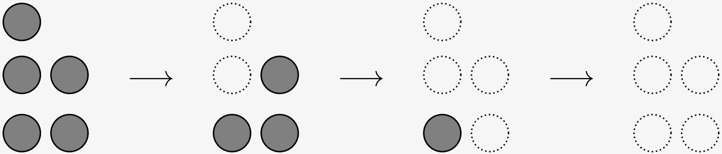
Nim

Setup

Nim is a classical two-player impartial non-random game played on stacks, or heaps, of chips. On each turn, a player removes any chip. Any chips that are no longer supported are removed with it. The player who takes the final chip wins, or alternatively, the player who can no longer move loses [Bou02].

Example 4

Example game sequence (the first player wins):



Relevance of Nim

Definition 5

A two-player impartial non-random game is said to be $*k$, or of **nimber k**, if the first player loses when the game is played consecutively to a k-stack of nim. Playing consecutively means that at each turn, the player has a choice whether to play in the original game or in the k-stack; the player who can no longer move loses.

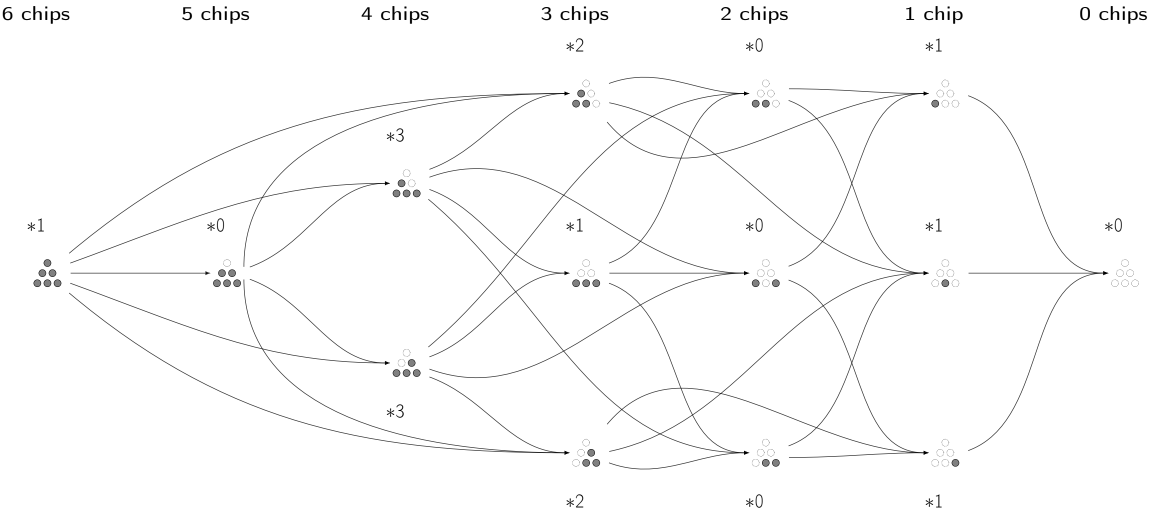
Theorem 6 (Sprague-Grundy Theorem)

[BCG01] Every two-player impartial finitely-terminating full-information game under normal play conditions has a nimber.

Theorem 7

Let X be a game with possible moves to states $\{X_1, \ldots, X_m\}$ of corresponding nimbers $*x_1, \ldots, *x_m$. Then the nimber of X, $*x$, is $x = \text{mex}\{x_1, \cdots, x_m\}$, where ***mex*** *denotes the* ***minimal excluded integer****.*

Computing Nimbers and Non-Isomorphic States



Strategies and P_n

n	1	2	3	4	5	6
P_n	$*1$	$*2$	$*1$	$*4$	$*1$	$*6$

n	7	8	9	10	11	12
P_n	$*1$	$*8$	$*1$	$*10$	$*1$	$*12$

Theorem 8

[Cur+22] In Pyranim, the first player has a winning strategy.

Proof. *The game begins in a removeable move; namely, taking the top chip off. (See: Chomp.)*

We conjecture that:

- If n is odd, then $P_n = *1$, and the winning move in the n-pyramid is to take the top chip.
- If n is even, then $P_n = *n$, and the winning move in the n-pyramid is to take all chips but the last one along one of the two diagonals.

Nim Addition

Theorem 9

Let X, Y be two games of nimbers $*x$ and $*y$. If X and Y are played consecutively, denoted as $X + Y$, then the nimber of $X + Y$, $*x + *y$, is $*(x \oplus y)$, where $\oplus$ denotes ***bitwise XOR****, also known as* ***nim addition****.*

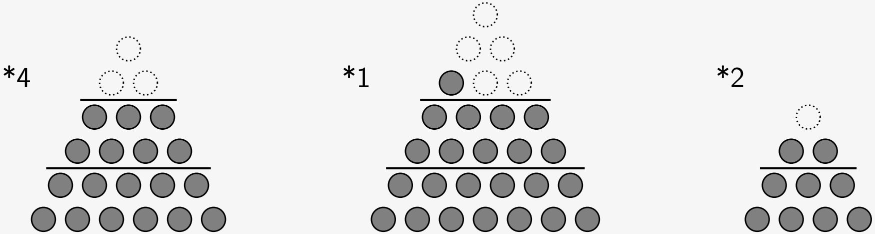
- This solves nim by allowing any player to find the next $*0$ positions.
- Nim addition is only (trivially) proven to work in Pyranim for adding completely disjoint states horizontally.
- We strongly suspect addition rules that allow determining the nimbers of special states; these may generalize to potentially arbitrary states.

Conjectured Pyranim Addition Rules

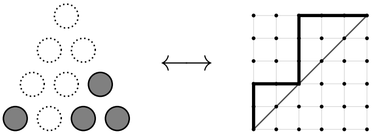
- Stacking $2k$ complete bases of odd support yields a $*0$ position.
- Stacking $2k$ complete bases of even support yields a $*(2k)$ position.
- If, with the previous setup, any chips are added *only on the layer above*, then the nimber is added the parity of the top chips ($*1$ if odd and $*0$ if even.)

Example 10

Left: $2 + 2 = 4$. Middle: $(0 + 0) \oplus 1 = 1$. Right: $2 \oplus (1 \oplus 1) = 2 \oplus 0 = 2$.



Basic Counting Results



$UUDDUUUDDD$

chip state $\longleftrightarrow$ Dyck path $\longleftrightarrow$ Dyck word.

Classical Fact (Catalan Numbers)

Let Dyck_{2n} *denote the set of Dyck paths of semilength* n*. Then*

$$\left|\mathrm{Dyck}_{2n}\right| = C_n := \frac{1}{n+1}\binom{2n}{n} \sim \frac{4^n}{n^{3/2}\sqrt{\pi}} \qquad [Sta15].$$

Computational Results: Isomorphisms

n	1	2	3	4	5	6
C_{n+1}	2	5	14	42	132	429
K_n	2	4	8	17	38	93
M_n	1	2	3	4	6	9
L_n	0	0	3	6	6	10

n	7	8	9	10	11	12
C_{n+1}	1430	4862	16796	58786	208012	742900
K_n	244	691	2080	6582	21628	73133
M_n	21	51	138	387	1134	3440
L_n	15	17	21	25	30	34

Bounds on K_n

Upper bound. Since C_{n+1} is *every* game state,

$$K_n \leq C_{n+1}.$$

To find a nontrivial bound on K_n, denote as I_n as the number of game states fixed under left-right reflection. Since we have the trivial bound of C_{n+1}, we see

$$K_n \leq C_{n+1} - \left(\frac{C_{n+1} - I_n}{2}\right) = \frac{C_{n+1} + I_n}{2}.$$

Furthermore, I_n, correlate to a combinatorial object called ballot paths (or Dyck prefixes) [BBS13], so we have the following closed form:

Theorem 11

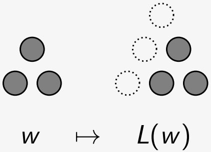
$$K_n \leq \frac{1}{2}\left(C_{n+1} + \binom{n+1}{\lfloor \frac{n+1}{2} \rfloor}\right).$$

Bounds on K_n

Lower bound. For a Dyck word w, define two extensions

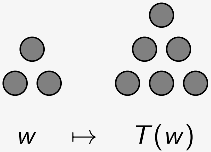
$$L, T : \mathcal{X}_n \longrightarrow \mathcal{X}_{n+1}, \qquad L(w) := UDw, \qquad T(w) := UwD.$$

L shifts the state one space to the right.



$w \mapsto L(w)$

T adds a full layer underneath.

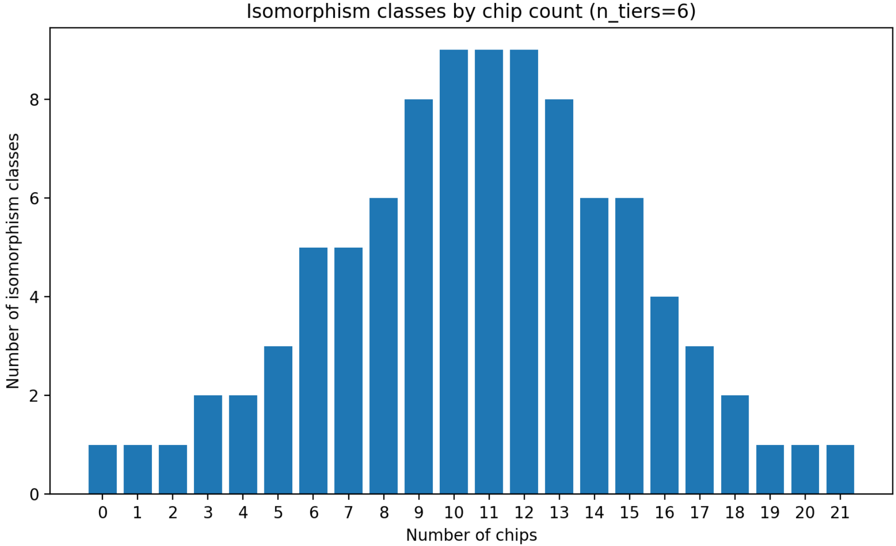


$w \mapsto T(w)$

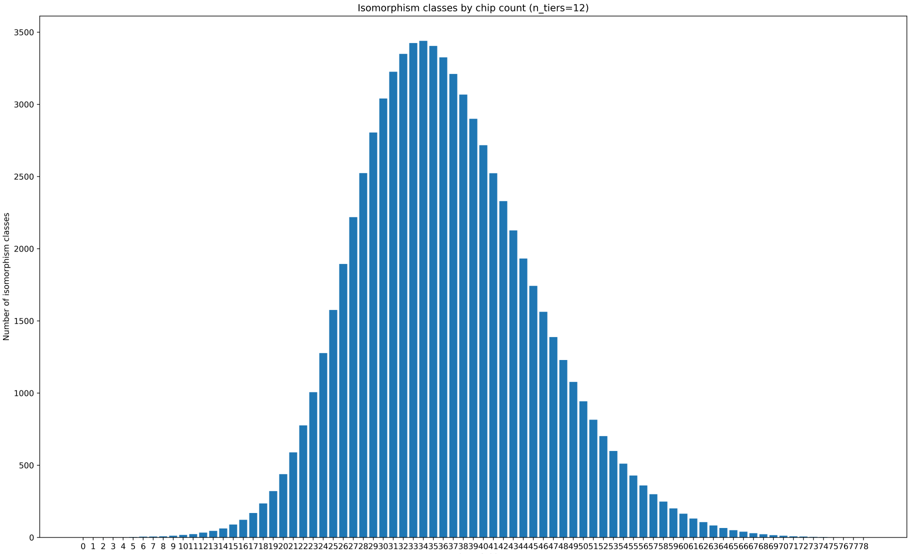
Theorem 12

The maps L and T descend to injections on isomorphism classes, and their images are disjoint. Hence every isomorphism class in tier $n-1$ produces two distinct classes in tier n, so $K_n \geq 2K_{n-1}$. Since $K_0 = 1$, this results in the lower bound $K_n \geq 2^n$.

Distribution of Equivalence Classes ($n = 6$)



Distribution of Equivalence Classes ($n = 12$)



A Gaussian Profile for $M_{n,k}$

Empirically, the sequence $k \mapsto M_{n,k}$ is unimodal and approaches a skewed normal-type profile.

Conjecture. $\exists$ a limiting density f, and a sequence of distribution centers μ_n, such that

$$\frac{n}{K_n} M_{n,\lfloor \mu_n + xn \rfloor} \to f(x), \qquad \text{w/} \qquad f(x) \approx \frac{1}{\sqrt{\pi}} e^{-x^2}.$$

i.e. f is approximately normal with variance $\frac{1}{2}$.

In particular, at the peak we obtain the approximation

$$M_n \approx \frac{K_n}{\sqrt{\pi}\, n}.$$

Bibliography

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Acknowledgments

We thank:

- Sergei Elizalde, who inspired our bijection to Dyck paths.
- Frank Sottile, who told us to solve the game before this conference, inspiring us to work on it relentlessly.
- Texas A&M University, the best university in the world. Go Tigers!